International Journal of Theoretical Physics, Vol. 13, No. 6 (1975), pp. 401–416

Gravitational Radiation Generated by the Gravitational Scattering of Two Stars†

CHRISTOPH MACHE and ECKART FREHLAND

Fachbereich Physik der Universität Konstanz, D-7750 Konstanz, Postfach 7733, Germany

Received: 8 *March* 1975

Abstract

Within the framework of general relativity, the gravitational scattering of two stars is considered with regard to the gravitational radiation effects. The angular and frequency dependence of the generated gravitational radiation is investigated by Fourier's analysis of the gravitational field in the radiation zone. The results of the numerical calculations show a strong directional dependence of the radiation in the plane of movement of the stars.

1. *Introduction*

Though up to now the positive results of Weber's gravitational wave experiments (e.g., Weber, 1969, 1970, 1972) could not be verified by other experimental groups (e.g., Tyson, 1973; Levine and Garwin, 1974), they have increased the interest in the general relativistic theory of gravitational radiation. The unfavorable energy balance of the experiments (rough estimations seem to indicate that per event an energy corresponding to the mass of the sun is radiated away) especially has been a main subject of theoretical investigations. Possibly the balance could be improved essentially if the radiation fields of the detected events show a strong directional dependence (footnote 1).

Independently of the results of the controversy about Weber's experiments, we believe that at the present state of theoretical investigation it is important to analyse and to understand in more detail the directional properties of the radiation fields generated by well defined sensible mass systems. In this paper we give a contribution to this program and discuss the gravitational scattering of two stars.

† Modified excerpt from the thesis of Christoph Mache, Konstanz, 1972.

[1] Indeed recent works indicate that this directional dependence exists in case the radiating objects are in fast motion ("synchroton radiation") (Doreshkevich *et al.*, 1972; Misner *et al.*, 1972; Fuchs, 1975).

In Section 2 we give a short introduction into the gravitational scattering process under the aspect of the occurring gravitational radiation damping. The gravitational energy radiated away, as calculated from pseudotensorial conservation laws in lowest approximation, may directly be translated into the loss of kinetic energy of the stars, a radiation damping effect, which at least in principle is observable. In Section 3 we summarize some aspects of the theory of gravitational radiation concerning Fourier's analysis of the radiation field.

The calculations throughout this paper are performed with the Landau-Lifshitz energy-momentum-pseudotensor (Landau-Lifshitz, 1967). The energy concept in general relativity has always been subject to controversies. We do not intend to give a contribution to this problem in this paper. All our results would be completely the same if we used other known energy expressions of the gravitational field (e.g., Einstein, 1918; Rosen, 1940; Kohler, 1953; Isaacson, 1968).

The result of the numerical calculations are given in Section 4. A main result is the strong directional dependence of the radiation in the plane of movement of the stars.

2. *Gravitational Radiation Damping as a Loss of Kinetic Energy*

Regard the conservation laws valid for some (pseudo-tensorial) energy expression $t^{\mu\nu}$ (e.g., the Landau-Lifshitz-pseudotensor) (footnote 2)

$$\frac{\partial}{\partial x^\nu}((-g)(T^{\mu\nu}+t^{\mu\nu}))=0 \tag{2.1}$$

By application of Gauss' law and time integration over the whole scattering process one gets the integral balance equation

$$\Delta E =: \left[\int_V (-g)(T^{44}+t^{44})\,d^3x\right]_{t=-\infty}^{t=+\infty} = \int_{t=-\infty}^{t=+\infty} c\,dt \int_F (-g)t^{4\alpha}\,df_a \tag{2.2}$$

In (2.2) the surface F enclosing the volume V is assumed to lie in the radiation zone.

The two basic assumptions under which our calculations are correct with sufficient accuracy are:

(a) The occurring velocities v_i (of the stars) are small compared with the velocity of light c (*low velocity assumption*)

$$\frac{v_i^2}{c^2} \ll 1 \tag{2.3}$$

(b) The gravitational interactions are weak (*weak field assumption*)

$$\frac{2mG}{c^2 d} \ll 1 \tag{2.4}$$

[2] Greek indices range and sum over 1, . . ., 4, latin indices over 1, 2, 3. Signature is chosen to be -2. $T^{\mu\nu}$: energy tensor of matter.

G: Newtons gravitational constant; m: mass of a star; d: characteristic length, i.e., distance of closest approach or radius of the stars.

As consequence of (a) and (b) the radiation (reaction) of the gravitational waves themselves, occurring in the rigorous theory because of the nonlinearity of field equations, may be neglected.

Under the assumptions (a) and (b) for calculations in the radiation zone the usual far-field approximation (see e.g., Landau and Lifshitz, 1967) may be applied. The course of the scattering process may roughly be described as follows

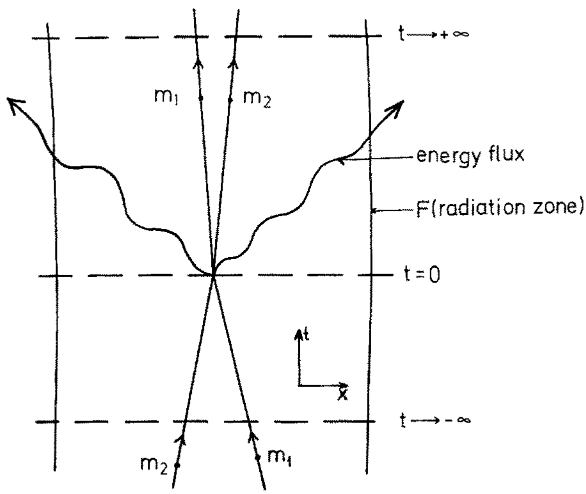

Figure 1—The course of the gravitational scattering of two stars in a spacetime diagram.

(see Figure 1): before and after the scattering the gravitational interactions between the two stars vanish. After a sufficiently long time the radiation parts of the gravitational field are flown away through the surface F in the radiation zone. We emphasize that this simplified description is applicable because our calculations are restricted to the lowest order of radiation damping effects, radiation reaction neglected.

The two volume integrals on the left-hand side (2.2), the difference of which we want to calculate, can in both cases be decomposed into two parts over V_1, the volume surrounding m_1, and V_2, the volume surrounding m_2, respectively.

We assume both stars to have constant proper masses m_1, m_2 and to be spherical symmetric (with constant radius) before and after the scattering process (spherical symmetry with respect to coordinates at rest relative to the stars).

Now we regard the star m_1 before the scattering (for $t \to -\infty$). Some observer in spacelike infinity being at rest relative to the coordinates x^a in which we calculate (e.g., center of mass system) will observe the star moving with uniform 3-velocity v_1 $(t = -\infty)$. Transformation from the coordinates x^a to coordinates $x_1{}^a$ being at rest relative to m_1 (before the scattering) is done by Lorentz-transförmation (velocity v_1 $(t = -\infty)$ between the two coordinate systems). After the scattering process we find a similar situation but possibly with another velocity v_1 $(t = +\infty)$.

Now consider the difference of the two integrals over V_1 for $t = -\infty$ and $+\infty$. If this difference is nonzero, the velocity of the star must have changed. For in case it would have not changed, the integrands in both integrals would be the same. Because of (a) and (b) the difference ΔE_1 of the two integrals must be small compared with the value E_1 of one of the integrals themselves

$$\left|\frac{\Delta E_1}{\underset{\pm\infty}{E_1}}\right| =: \left| \frac{\left[\int_{V_1} (-g)(T^{44} + t^{44})\, d^3x \right]_{t=-\infty}^{t=+\infty}}{\left[\int_{V_1} (-g)(T^{44} + t^{44})\, d^3x \right]_{t=\pm\infty}} \right| \ll 1 \tag{2.5}$$

The integrals $\underset{+\infty}{E_1}, \underset{-\infty}{E_1}$ may be decomposed into terms of different orders of magnitude with the dominant terms

$$\frac{m_1}{\sqrt{\left(1 - \dfrac{v_1{}^2(t = -\infty)}{c^2}\right)}}, \qquad \frac{m_1}{\sqrt{\left(1 - \dfrac{v_1{}^2\,(t = +\infty)}{c^2}\right)}}$$

respectively. The magnitude of the other terms is determined by the gravitational eigenfield of the star and small compared with the dominant terms. Their contributions to the difference ΔE_1 are small compared with the contribution of the dominant terms. For m_2 we can do corresponding considerations. For low velocities v holds

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$$

Hence from (2.2) we get finally

$$\Delta E = \frac{m_1}{2}\left(v_1{}^2(t = +\infty) - v_1{}^2(t = -\infty)\right) + \frac{m_2}{2}\left(v_2{}^2(t = +\infty) - v_2{}^2(t = -\infty)\right) \tag{2.6}$$

The velocities $v_i(t = \pm\infty)$ are the initial (final) velocities as measured by an observer in spacelike infinity. Therefore the total radiated energy ΔE is (in lowest approximation) equal to the loss of kinetic energy of the stars and can be observed as a decrease of velocities of the stars.

3. *Fourier's Analysis of Gravitational Radiation Fields*

The spacetime is assumed to be asymptotically flat. There exist coordinates in which the metric $g_{\mu\nu}$ becomes Minkowskian in the spacelike infinity. Furthermore one can introduce coordinates obeying the (De-Donder) condition

$$\frac{\partial \psi^{\mu\nu}}{\partial x^{\nu}} = 0, \qquad (h^{\mu\nu} - \tfrac{1}{2} h \eta^{\mu\nu}) = \psi^{\mu\nu}, \qquad h = h_{\alpha\beta}\eta^{\alpha\beta} \tag{3.1}$$

where in the far field zone the deviations $h^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$ from the Minkowski metric $\eta^{\mu\nu}$ are small. In the far field the outgoing gravitational waves may be regarded as being plane fronted over finite regions. In these finite regions one can introduce special De-Donder coordinates, in which the gravitational potentials $h^{\mu\nu}$ of the plane fronted waves are (see e.g., Landau and Lifshitz, 1967)

$$h^{22} = -h^{33}, \qquad h^{23} = h^{32} \tag{3.2}$$

and $h^{\mu\nu} = 0$ in all other components (see footnote 3). The waves are traveling in the direction of the x^1-axis. The energy flux t^{01} in the x^1-direction as calculated with the Landau-Lifshitz pseudotensor $t^{\mu\nu}$ is

$$t^{01} = \frac{c^2}{16\pi G}\left[\left(\frac{1}{2}\frac{\partial(h_{22} - h_{33})}{\partial t}\right)^2 + \left(\frac{\partial h_{23}}{\partial t}\right)^2\right] \tag{3.3}$$

As is well known and as may be verified by direct calculation from the pseudotensor $t^{\mu\nu}$ the expression (3.3) for the energy-flux of a (weak) plane wave is invariant under small (gauge) transformations within the group of coordinates satisfying the De-Donder condition (3.1). Furthermore $t^{\mu\nu}$ is invariant under linear transformations. This justifies that in our explicit calculations in Section 4 we can use the Landau-Lifshitz far-field approximation in its usual form, i.e. in coordinates satisfying (3.1) but generally not (3.2). The numerical results below are independent of the special choice of coordinates.

In the radiation field we make a Fourier's analysis of the potentials $h^{\mu\nu}$

$$\tilde{h}^{\mu\nu}(\omega) = \int_{-\infty}^{+\infty} h^{\mu\nu} \exp(i\omega t)\, dt \tag{3.4}$$

Herewith we get as well the Fourier spectrum of the first and further derivatives of $h^{\mu\nu}$

$$\widetilde{\left(\frac{\partial^n h^{\mu\nu}}{\partial t^n}\right)}(\omega) = (-i\omega)^n \cdot \tilde{h}^{\mu\nu}(\omega) \tag{3.5}$$

[3] For simplicity in this chapter we use the special coordinates given by (3.2). Because of the bilinear structure of $t^{\mu\nu}$ Fouriers' analysis in other coordinates is done analogously.

The harmonic spectrum of the energy flux is in analogy to electrodynamics

$$\tilde{t}^{\,01}(\omega) = \frac{c^2}{8\pi^2 G}\left(\left|-\frac{i\omega}{2}(\tilde{h}_{22}(\omega) - \tilde{h}_{33}(\omega)\right|^2 + |-i\omega\tilde{h}_{23}(\omega)|^2\right) \tag{3.6}$$

Integration over positive ω and over a sphere in the radiation zone enclosing the radiating source region yields the totally radiated gravitational energy ΔE.

Einstein's pseudotensor $t_\mu{}^\nu$ as well as the Isaacson stress-energy-tensor (Isaacson, 1968) and energy tensors derived from a theory of double measure in first approximation (Rosen, 1940; Kohler, 1953; Westpfahl, 1967) yield the same results. For the Einstein pseudotensor, which has the same invariance properties as mentioned above for $t^{\mu\nu}$, the energy flux of a (weak) plane wave agrees with (3.3). In the other cases one easily shows the equality of the harmonic spectra with (3.6).

We want to emphasize that the given Fourier's analysis of the energy flux in the radiation zone is valid in asymptotically flat spacetimes and *independent* of the nature of the events which generate the gravitational waves. These events may be extreme relativistic or weak field interactions, as in our example.

We mention that the given Fourier's analysis of the radiation fields yields a measure for the effect of the radiation on Weber-type gravitational wave detectors, which for simplification we assume to be a linear harmonic oscillator (two masses coupled with a spring). It can be shown (Weber, 1962; Frehland, 1971; Papapetrou, 1972; Maugin, 1973) that the driving force K^a for the oscillator is determined by the components R^a_{4b4} of the Riemann-tensor $R^\alpha_{\beta\gamma\delta}$ of the gravitational field

$$\frac{K^a}{m} = c^2 R^a_{4b4} r^b \tag{3.7}$$

where r^b is the position vector with reference to the center of mass of the oscillator. In the radiation zone $\tilde{R}^a_{4b4}(\omega)$ is according to (3.2) and (3.5) given by

$$\tilde{R}^a_{4b4}(\omega) = -\frac{\omega^2}{2c^2}\tilde{h}_{ab}(\omega) \tag{3.8}$$

The energy flux $t^{01}(\omega)$ is a measure for the excitation of the oscillator. In case the oscillator is oscillating vertical to the propagation direction of a circularly polarized gravitational wave the mean value $\bar{K}(\omega)$ of the spectral force density exciting the linear oscillator

$$\frac{\bar{K}(\omega)}{m} = \mu \,.\, \frac{2\pi \,.\, \omega\ \sqrt{(G)}}{c}\ \sqrt{(t^{01}(\omega))} \tag{3.9}$$

For linearly polarized waves the factor μ varies between 1 and 0 according to the position of the oscillator.

4. *Detailed Calculations and Numerical Results*

We assume the stars to move on Kepler hyperbolic orbits with eccentricities $e > 1$. First we shall compute the angular distribution of the energy flux. From this result we then get the total energy loss by integration over all directions. Then we shall investigate the angular distribution of the harmonic spectrum of the energy flux and the total energy spectrum.

The radial energy flux density Φ radiated from a localized system of masses moving in the x, y-plane and calculated with the Landau-Lifshitz pseudotensor is in lowest approximation

$$\begin{aligned}\Phi = \frac{c^4}{16\pi G}\{&\tfrac{1}{4}\psi^{11}_{14}\psi^{11}_{14}(1-\cos^2\varphi.\sin^2\vartheta)^2 + \\ &+ \psi^{11}_{14}\psi^{12}_{14}(-\cos\varphi.\sin\varphi\sin^2\vartheta + \cos^3\varphi\sin\varphi\sin^4\vartheta) + \\ &+ \tfrac{1}{2}\psi^{11}_{14}\psi^{22}_{14}(-\cos^2\vartheta + \cos^2\varphi\sin^2\varphi\sin^4\vartheta) + \\ &+ \psi^{12}_{14}\psi^{22}_{14}(-\cos\varphi\sin\varphi\sin^2\vartheta + \cos\varphi\sin^3\varphi\sin^4\vartheta) + \\ &+ \tfrac{1}{4}\psi^{22}_{14}\psi^{22}_{14}(1-\sin^2\varphi.\sin^2\vartheta)^2\}\end{aligned} \tag{4.1}$$

In (4.1) the derivatives of the gravitational potentials $\psi^{4\alpha}$ are all expressed by ψ^{ab}_{14} using the De-Donder condition (3.1). The gravitational potentials ψ^{ab} are given by the far-field approximation (Landau and Lifshitz, 1967):

$$\psi^{ab} = \frac{2G}{c^4 R_0}\cdot\frac{\partial^2}{\partial t^2}\int_V \rho_0 x^a x^b \, dV \tag{4.2}$$

where R_0 is the distance from the radiating system and ρ_0 mass density. The equations of the orbit are:

$$\begin{aligned} x &= \frac{G(m_1+m_2)}{v_0^2}(e - ch\xi) \\ y &= \frac{G(m_1+m_2)}{v_0^2}(e^2-1)^{1/2}\, sh\xi \\ t &= \frac{G(m_1+m_2)}{v_0^3}(e\,.sh\xi - \xi)\end{aligned} \tag{4.3a}$$

where m_1, m_2 are the masses of the stars, m their reduced mass, v_0 relative velocity before the scattering process and e the eccentricity of the orbit which is defined by

$$e = \left(1 + \frac{v_0^4\,.\rho^2}{G^2(m_1+m_2)^2}\right)^{1/2} \tag{4.3b}$$

ρ is the impact parameter.

We use the abbreviations

$$\int_n = \int_{-\infty}^{+\infty} \frac{dt}{\left(cht - \frac{1}{e}\right)^n} \tag{4.4}$$

Then the total energy dE radiated into a solid angle $d\Omega$ in a direction determined by the polar angles ϑ, φ is (in the center of mass system) with (4.1)–(4.4)

$$dE = \left(\int_{-\infty}^{+\infty} \Phi\, dt\right) . R_0^2\, d\Omega$$

$$= \frac{m^2 v_0^7}{4\pi c^5 (m_1 + m_2) e^7} \Bigg\{ \Bigg[\int_3 + \left(4e - \frac{2}{e}\right) \int_4 - \left(2 - \frac{9}{e^2}\right) \int_5 -$$

$$- \left(\frac{12}{e} - \frac{4}{e^3}\right)(e^2-1) \int_6 + \left(\frac{9}{e^2} - \frac{31}{e^4}\right)(e^2-1)^3 \int_7 + \left(\frac{30}{e^5}\right)(e^2-1)^4 \int_8 -$$

$$- \left(\frac{9}{e^6}\right)(e^2-1)^5 \int_9 \Bigg] . (1 - \cos^2\varphi . \sin^2\vartheta)^2 +$$

$$+ (e^2-1) \Bigg[2\int_3 + \left(\frac{4}{e}\right)(e^2-1) \int_4 - \left(6e^2 - 16 + \frac{18}{e^2}\right) \int_5 + \left(\frac{8}{e^3}\right)(e^2-1) \int_6 -$$

$$- \left(\frac{12}{e^2} - \frac{62}{e^4}\right)(e^2-1) \int_7 - \left(\frac{60}{e^5}\right)(e^2-1)^3 \int_8 + \left(\frac{18}{e^6}\right)(e^2-1) \int_9 \Bigg] .$$

$$. (-\cos^2\vartheta + \cos^2\varphi \sin^2\varphi \sin^4\vartheta) +$$

$$+ (e^2-1) \Bigg[4\int_3 + \left(\frac{8}{e}\right)(e^2-1) \int_4 + \left(4 + \frac{36}{e^2}\right)(e^2-1) \int_5 + \left(\frac{16}{e^3}\right)(e^2-1)^2 \int_6 -$$

$$- \left(\frac{24}{e^2} - \frac{124}{e^4}\right)(e^2-1)^2 \int_7 - \left(\frac{120}{e^5}\right)(e^2-1)^3 \int_8 + \left(\frac{36}{e^6}\right)(e^2-1)^4 \int_9 \Bigg] .$$

$$. (\cos^2\vartheta + \cos^2\varphi . \sin^2\varphi . \sin^4\vartheta) +$$

$$+ (e^2-1)^2 \Bigg[\int_3 - \left(\frac{2}{e}\right) \int_4 + \left(5 - \frac{9}{e^2}\right) \int_5 + \left(\frac{4}{e} + \frac{4}{e^3}\right) \int_6 +$$

$$+ \left(\frac{3}{e^2} - \frac{31}{e^4}\right)(e^2-1) \int_7 + \left(\frac{30}{e^5}\right)(e^2-1)^2 \int_8 - \left(\frac{9}{e^6}\right)(e^2-1)^3 \int_9 \Bigg] .$$

$$. (1 - \sin^2\varphi . \sin^2\vartheta)^2 \Bigg\} d\Omega \tag{4.5}$$

Integration of dE over a sphere yields the total energy ΔE radiated away during the scattering process:

$$\Delta E = \frac{2m^2 v_0{}^7}{15C^5(m_1 + m_2)} \left[\frac{673e^2 + 602}{3(e^2 - 1)^3} + \frac{37e^4 + 292e^2 + 96}{(e^2 - 1)^{7/2}} \left(\text{arc sin} \frac{1}{e} + \frac{\pi}{2} \right) \right] \tag{4.6}$$

According to Section 2 ΔE is equal to the loss of kinetic energy of the stars. (4.6) agrees with a result recently given by Hansen (1972) except that, upon repeating his calculations, we find a term 673 e^2, where it also appears in the present caclulation, in place of Hansen's 457 e^2.

For values $e \gg 1$ (4.6) yields approximately with (4.3b)

$$\Delta E \approx \frac{37}{15} \pi \frac{G^3}{\rho^3} m_1{}^2 \,.\, m_2{}^2 v_0 \tag{4.7}$$

This result is in agreement to Peters (1970), who has discussed the gravitational scattering of a small mass passing a large mass for arbitrary velocities within a perturbational approach and straight uniform motion of the small mass as first approximation.

The quantitative evaluation of (4.5) and (4.6) is given in Figures 2 and 3a, b, c.

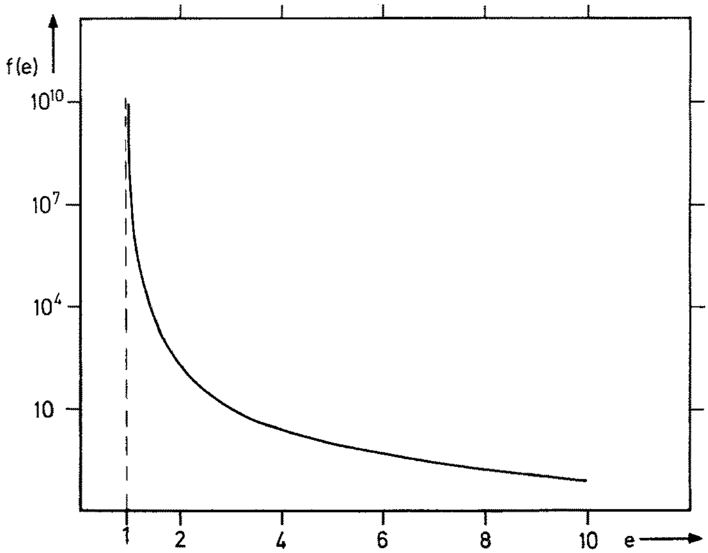

Figure 2–Total loss of kinetic energy $-\Delta E$ as a function of the eccentricity e. According to (4.6), $-\Delta E$ is given by $f(e)$ through the relation $-\Delta E = f(e)$. $(2m^2 v_0{}^7/15c^5(m_1 + m_2))$.

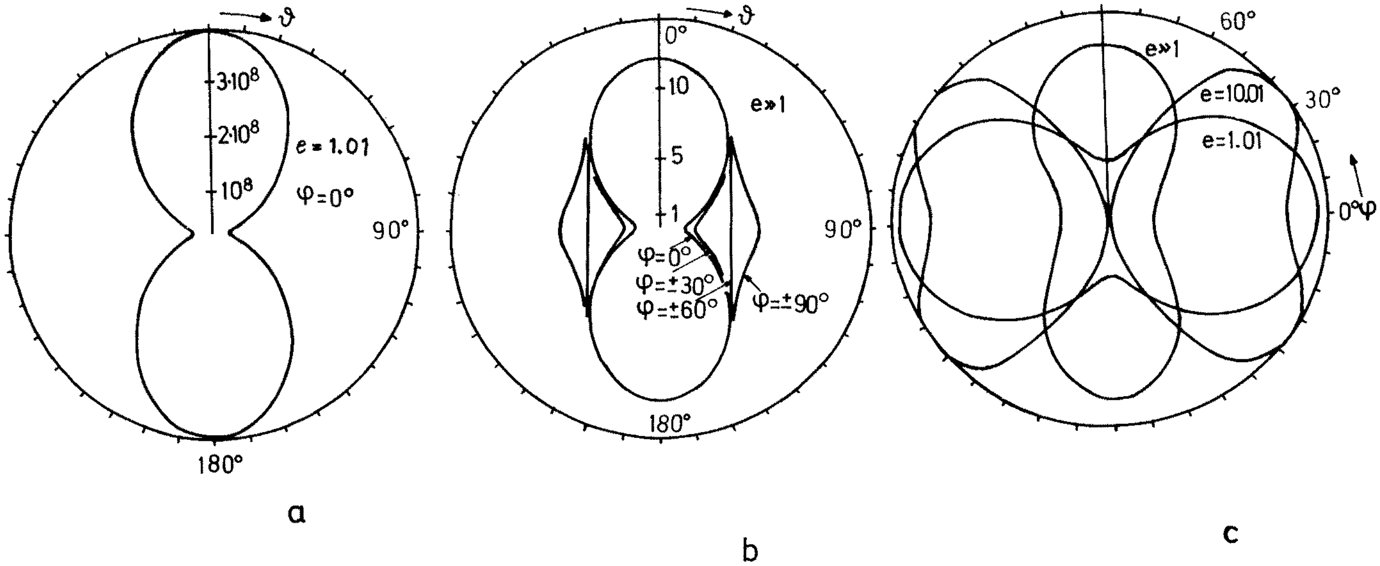

Figure 3—Directional dependence of total energy dE radiated into the solid angle $d\Omega$ according to (4.5) (in arbitrary units). (a) Directional dependence for eccentricity $e = 1.01$ in meridianal plane $\varphi = 0°$. In the other meridianal planes φ = constant the dependence is similar. (b) Directional dependence for eccentricities $e \gg 1$ in different meridianal planes φ = constant. (c) Angular distribution of dE in the x, y-plane ($\vartheta = 90°$) for different eccentricities e.

The angular distribution of the harmonic spectrum of the radiated energy is given by replacing in (4.1) the products of the time derivatives of the potentials ψ^{ab} by the products of their Fourier-transformations in the following way (cf. (3.5))

$$\frac{\partial}{\partial t}\psi^{ab}\cdot\frac{\partial}{\partial t}\psi^{cd}\longrightarrow\frac{\omega^2}{\pi}Re(\tilde{\psi}^{ab}(\omega)\cdot\tilde{\psi}^{cd}*(\omega))$$

where

$$\tilde{\psi}^{ab}(\omega)=\int_{-\infty}^{+\infty}\psi^{ab}\exp(i\omega t)\,dt$$

and $\tilde{\psi}^{ab}*$ is the conjugate complex of $\tilde{\psi}^{ab}$. From (4.1) we get for the energy loss per solid angle unit

$$d\left(\frac{dE}{d\Omega}(\omega,\vartheta,\varphi)\right)=\frac{c^3R_0{}^2\omega^2}{16\pi G}\left[\frac{1}{4}(\tilde{\psi}^{11}(\omega))^2\cdot(1-\cos^2\varphi\,\sin^2\vartheta)^2+\right.$$
$$+\tfrac{1}{2}\tilde{\psi}^{11}(\omega)\cdot\tilde{\psi}^{22}(\omega)(-\cos^2\vartheta+\cos^2\varphi\sin^2\varphi\sin^4\vartheta)+$$
$$+|\tilde{\psi}^{12}(\omega)|^2(\cos^2\vartheta+\cos^2\varphi\sin^2\varphi\sin^4\vartheta)+$$
$$\left.+(\tilde{\psi}^{22}(\omega))^2(1-\sin^2\varphi\sin^2\vartheta)^2\right]d\omega \qquad (4.8)$$

The Fourier-transformations of ψ^{11}, ψ^{12}, ψ^{22} yield

$$\tilde{\psi}^{11}(\omega)=-\frac{4G^2m_1m_2}{c^4R_0v_0}\int_{-\infty}^{+\infty}\left(\frac{1}{e^2}+\frac{\left(\frac{1}{e^3}\right)(e^2-1)}{\left(cht-\frac{1}{e}\right)}-\right.$$
$$\left.-\frac{\left(\frac{1}{e^4}(e^2-1)^2\right)}{\left(cht-\frac{1}{e}\right)^2}\right)\exp(i\omega'(esht-t))\,dt \qquad (4.9a)$$

$$\tilde{\psi}^{12}(\omega)=\frac{4G^2m_1m_2}{c^4R_0v_0}\cdot\frac{i}{\omega'}\int_{-\infty}^{+\infty}\left(-\frac{\frac{1}{e^3}}{\left(cht-\frac{1}{e}\right)}-\frac{\frac{1}{e^4}}{\left(cht-\frac{1}{e}\right)^2}-\right.$$
$$\left.-\frac{\left(\frac{5}{e^5}\right)(e^2-1)}{\left(cht-\frac{1}{e}\right)^3}+\frac{\left(\frac{3}{e^6}\right)(e^2-1)^2}{\left(cht-\frac{1}{e}\right)^4}\right)\exp(i\omega'(esht-t))\,dt \qquad (4.9b)$$

$$\tilde{\psi}^{22}(\omega) = -\frac{4G^2 m^1 m_2}{c^4 R_0 v_0}(e^2-1)\int_{-\infty}^{+\infty}\left(\frac{1}{e^2} - \frac{\frac{1}{e^3}}{\left(cht - \frac{1}{e}\right)} + \right.$$

$$\left. + \frac{\left(\frac{1}{e^4}\right)(e^2-1)}{\left(cht - \frac{1}{e}\right)^2}\right)\exp\left(i\omega'(esht - t)\right)dt \qquad (4.9c)$$

with the dimensionless "frequency"

$$\omega' = \frac{\omega \,.\, G \,.\, (m_1 + m_2)}{{v_0}^3} \qquad (4.9d)$$

The numerical evaluation of these integrals gives according to (4.8) the angular distribution of the harmonic spectrum of the radiated energy. The quantitative results are shown in Figures 4 and 5. The radiation in direction of the z-axis is stronger in at least one order of magnitude than in the plane of movement for the harmonic spectrum as well as for the totally radiated energy.

The spectrum of total energy is given by integration of (4.8) over the total solid angle:

$$dE(\omega) = \frac{c^3 {R_0}^2 \omega^2}{30\pi G} \,.\, \{|\tilde{\psi}^{11}(\omega)|^2 - \tilde{\psi}^{11}(\omega)\tilde{\psi}^{22}(\omega) +$$

$$+ 3|\tilde{\psi}^{12}(\omega)|^2 + |\tilde{\psi}^{22}(\omega)|^2\}\, d\omega \qquad (4.10)$$

The spectrum is shown in Figure 6 for different eccentricities e. The fact that for all eccentricities the energy distribution $E(\omega')$ reaches a maximum (at $\omega' = \omega'_{max}$) makes possible the definition of a characteristic collision time t_c. From ω'_{max}, which is to be taken from Figure 6, we get ω_{max}

$$\omega_{max} \approx \frac{2\pi}{t_c} \approx \frac{\omega'_{max}}{G(m_1 + m_2)}{v_0}^3$$

and hence

$$t_c = \frac{2\pi}{\omega'_{max}} \frac{G(m_1 + m_2)}{{v_0}^3} \qquad (4.11)$$

We mention that a characteristic collision time might also be determined from the time-course of the totally radiated power $\dot{E}$. $\dot{E}$ can be calculated by inte-

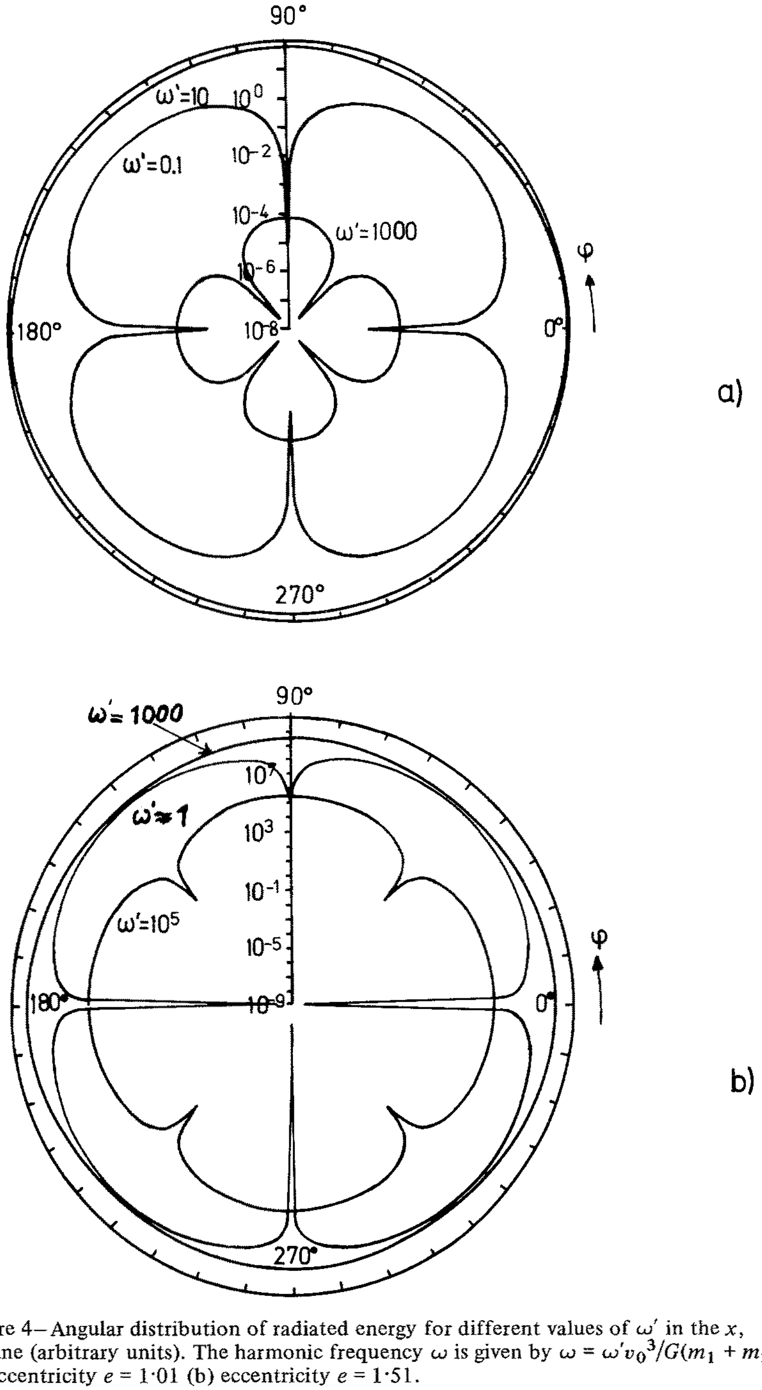

Figure 4—Angular distribution of radiated energy for different values of ω' in the x, y-plane (arbitrary units). The harmonic frequency ω is given by $\omega = \omega' v_0{}^3/G(m_1 + m_2)$ (a) eccentricity $e = 1{\cdot}01$ (b) eccentricity $e = 1{\cdot}51$.

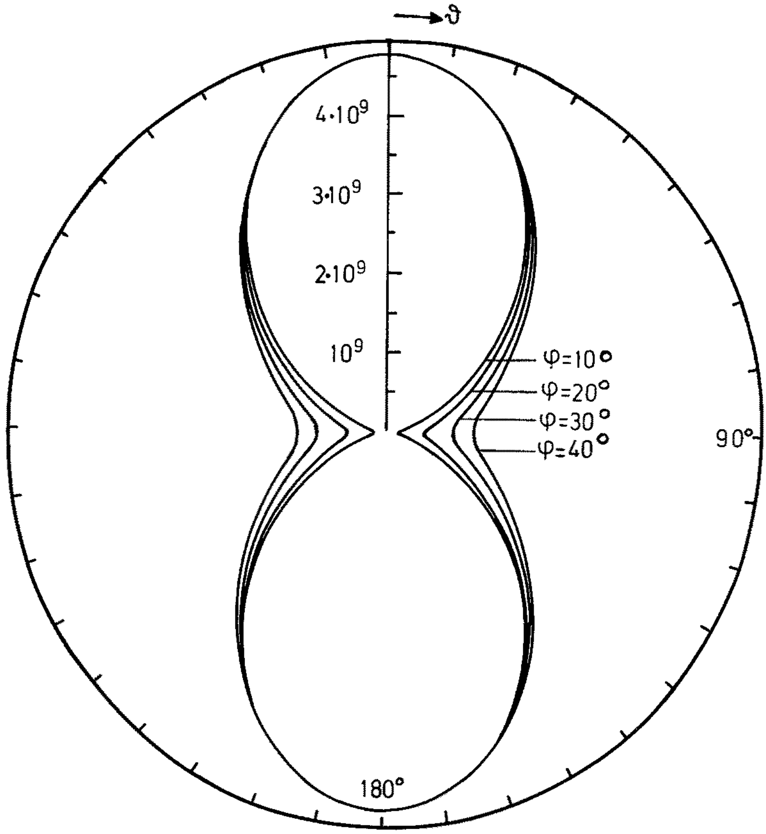

Figure 5—Directional dependence of the radiated energy at $\omega' = 0{\cdot}1$ in different meridianal planes. For other frequencies ω' the qualitative shape is similar.

gration of (4.1) over the total solid angle or using the well known formula for quadrupole radiation (see Landau and Lifshitz, 1967). Indeed quantitative calculations of the half-life time of $\dot{E}$ yield agreement with t_c within a factor < 2.

5. *Discussion*

The results for the total radiated energy presented in Figure 2 and (4.6) include the interesting case of gravitational capture of the stars due to gravitational radiation damping ($2\Delta E/mv_0^2 > 1$). In this case the generated double star system continues to radiate and the total radiated energy increases.

Figures 3a and 5 show that the principal part of energy is radiated in a direction normal to the plane of movement. On the other hand, according to Figure 3c, the radiation within the plane of movement is strongly dependent on the eccentricity e. For great eccentricities $e \gg 1$ (i.e., nearly straight-lined

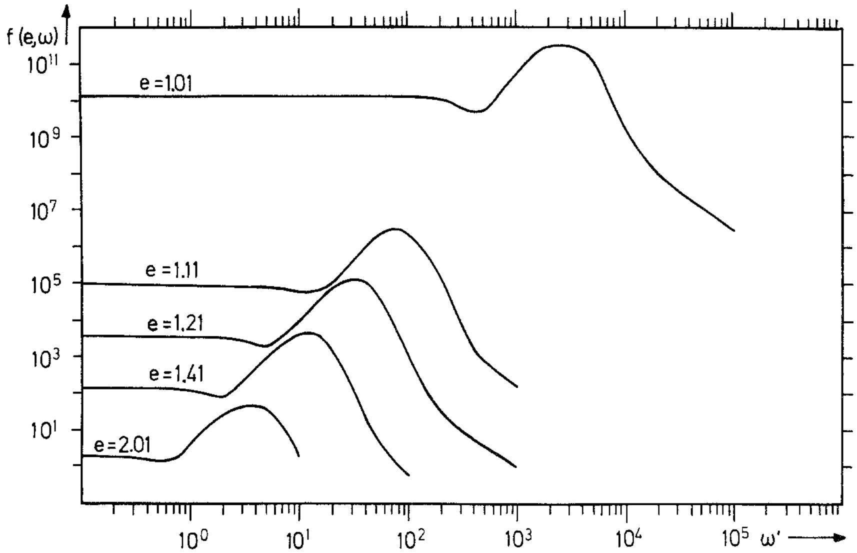

Figure 6—Harmonic spectrum of the total energy loss for different eccentricities e as a function of ω'. The total energy loss $-dE$ in the interval $[\omega', \omega' + d\omega']$ is given by $-dE = f(e, \omega')\,(8Gm^2v_0^4/15\pi^2c^5)\,d\omega'$.

orbits of the stars) the principal part is radiated in the direction of movement of the stars. This directional dependence indicates the possibility of improvement of the energy balance mentioned in the introduction. But naturally an energy flux (on earth) of 10^4 erg/sec cm^2, as required by Weber's experiments, cannot be reached under the assumptions (2.3) and (2.4) of low velocities and weak fields. If we assume two neutron stars with sun masses moving in the galactic center with an intrinsic relative velocity of 10^7 cm/sec and a minimum distance of $4 \, . \, 10^6$ cm, the maximum energy flux on earth is at least 5 orders of magnitude too small in case the plane of movement lies in the galactic plane.

Acknowledgments

We thank Prof. H. Dehnen for interesting discussions on this subject. One of us (E. F.) gratefully acknowledges the support of the Deutsche Forschungsgemeinschaft. The numerical calculations have been performed at the TR 86 of the Rechenzentrum der Universität Konstanz.

References

Doreshkevich, A. G., Novikov, J. D., and Polnarev, A. G. (1972). *Soviet Physics JETP*, **63**, 1538.

Einstein, A. (1918). *Sitz. Ber. Pr. Akad. d. Wiss.* 448.

Frehland, E. (1971). *Physical Letters*, **35A**, 432.

Fuchs, R. (1975). (To be published).

Isaacson, R. A. (1968). *Physical Review*, **166**, 1263.
Kohler, M. (1953). *Z. Phys.* **134**, 306.
Landau, L. D. and Lifshitz, E. M. (1967). *Klassische Feldtheorie*, Akademie-Verlag Berlin.
Levine, J. L. and Garwin, R. L. (1974). *Physical Review Letters*, **33**, 797.
Hansen, R. O., (1972). *Physical Review*, **D5**, 1021.
Maugin, G. A., (1973). *General Relativity and Gravitation*, **4**, 241.
Misner, C. W., Breuer, R. A., Brill, D. R., Chrzanoswki, P. L., Hughes, H. G., and Pereira, C. M. (1972). *Physical Review Letters*, 28, 998.
Papapetrou, A. (1972). *Ann. Inst. Poincare*, **16**, 63.
Peters, P. C. (1970). *Physical Review*, **D6**, 1559.
Rosen, N. (1940). *Physical Review*, 57, 147.
Tyson, J. A. (1973). *Physical Review Letters*, **31**, 326.
Weber, J. (1962). *General Relativity and Gravitational Waves,* Interscience Publishers, New York.
Weber, J. (1969), *Physical Review Letters*, **22**, 1320.
Weber, J. (1970). *Physical Review Letters*, **24**, 276.
Weber, J. (1972). *General Relativity and Gravitation*, **3**, 59.
Westpfahl, K. (1967). *Fortschr. d. Phys.* **15**, 309.